Journal of Sound and Vibration 329 (2010) 2707–2715

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Rapid Communication

Vibration of a string wrapping and unwrapping around an obstacle

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ARTICLE INFO

Article history:
Received 6 August 2009
Received in revised form
28 January 2010
Accepted 3 February 2010
Handling editor: M.P. Cartmell

ABSTRACT

The problem of a string vibrating against a smooth obstacle is investigated in this paper. The obstacle is located at one of the boundaries and the string is assumed to wrap and unwrap around the obstacle during vibration. The wrapping of the obstacle is modeled by a series of perfectly inelastic collisions between the obstacle and adjacent segments of the string and unwrapping is assumed to be energy conserving. The geometry of the string is determined iteratively starting from an initial configuration where the string is vibrating in a single mode and is not in contact with the obstacle. The obstacle can be regarded as a passive mechanism for vibration suppression in which the energy lost during each cycle of oscillation depends on the energy content of the string at the beginning of the cycle. Numerical simulation results are provided for the string vibrating in different modes for circular- and elliptic-shaped obstacles. The loss of energy is found to be greater for higher modes of oscillation and for obstacles that induce greater length of wrapping.

1. Introduction

The dynamics of vibrating strings has been a subject of study for a very long time but the motion of strings vibrating against obstacles appeared in the technical literature relatively recently. Early work on this problem can be credited to Citrini [1] who considered point-shaped obstacles. The element of string that comes in contact with point-shaped obstacles can be assumed to be massless and hence the energy of the string, in the absence of damping, was assumed to remain conserved. Amerio [2] investigated the motion of a string vibrating against a rigid wall, parallel to the position of the string at rest. The motion of the string in the presence of the unilateral constraint was posed as a problem in impact. The nature of the impact was assumed to be elastic and the problem was formulated based on conservation of energy of the string. A number of other researchers have also based their work on the premise of energy conservation of the string. These include Schatzman [3], who investigated the existence and uniqueness of solutions for concave obstacles and Haraux and Cabannes [4], who established almost-periodic nature of solutions for straight and fixed obstacles.

In 1982, Burridge et al. [5] investigated the vibration of the sitar, an Indian stringed instrument. The sitar differs from the Western stringed instruments in that the bridge across which the strings pass form a broad support, rather than a well-defined edge. During vibration, the sitar string wraps and unwraps around the gentle slope of the bridge and the length of the vibrating part of the string varies during oscillation [5]. Burridge et al. [5] modeled the impact of the string with the bridge as perfectly inelastic, discarding the assumption of energy conservation of the string. Subsequently, Bamberger and Schatzman [6] proved the existence of solutions which do not conserve energy with arbitrary obstacles and Ahn [7] claimed energy loss of the string vibrating against flat obstacles. In conformity with earlier work by Citrini [1], Ahn [7] also

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doi:10.1016/j.jsvi.2010.02.006

showed that energy remains conserved for highly peaked obstacles. Other work on string vibration against obstacles includes discretization [8] and finite difference methods [9] for numerical simulation, and study of nonlinear effects of varying amplitude and gravity [10] on extensible and non-extensible cables.

In this paper we investigate the vibration of a string against an obstacle located at its boundary. Similar to the work by Burridge et al. [5], we assume the string to wrap and unwrap around the obstacle during each oscillation. The impact of the string during wrapping is assumed to be perfectly inelastic and the obstacle is implicitly assumed to be convex. The assumption of convexity of the obstacle is both convenient and practical. Assuming that the string vibrates in a single mode at all times, it is shown that energy loss is higher for higher modes of oscillation. Although oscillation in a higher mode results in less wrapping, higher energy loss results from higher kinetic energy of the string during impact. The obstacle constrains the motion of the string and in this regard the mechanism for energy loss is a continuous-system version of the energy dissipation methodology proposed for finite degree-of-freedom systems by Issa et al. [11]. Since the energy of the string decreases even in the absence of damping, the obstacle can be regarded as a passive mechanism for vibration suppression and control. It should be noted that vibration control is not the focus of this paper and efficient methods can be developed using optimal control and/or feedback control methods, such as those proposed by Liu [12] and Shahruz and Narasimha [13].

This paper is organized as follows. A formal problem statement and a list of the assumptions made in our analysis is provided in Section 2. In Section 3 we present our analytical model for computing the geometry of the string as it wraps and unwraps around the obstacle during oscillation. In Section 4 we provide simulation results for percentage energy loss and length of wrapping during each cycle of oscillation for different modes with circular- and elliptic-shaped obstacles. Using numerical simulations, we show in Section 5 that percentage energy loss can be increased significantly by changing the orientation of the obstacle. Section 6 provides concluding remarks.

2. Problem statement and assumptions

Consider a string vibrating against an obstacle placed at one of its boundaries, as shown in Fig. 1. We investigate energy dissipation in the string under the following assumptions:

A1. The obstacle is rigid and has the following geometry:

$$y = f(x), \quad y(0) = 0, \quad \left[\frac{dy}{dx}\right]_{x=0} = 0 \tag{1}$$

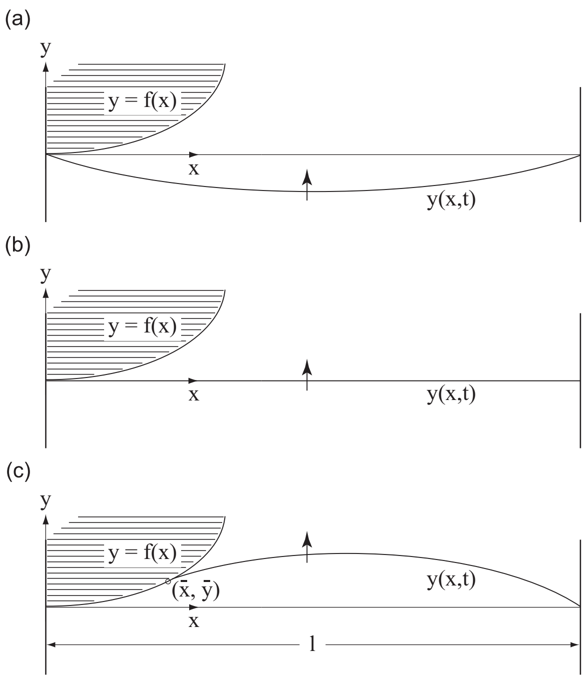

Fig. 1. A string vibrating against an obstacle is shown in three configurations: (a) the string has both potential and kinetic energy and is not in contact with the obstacle, (b) the string has kinetic energy but no potential energy and is not in contact with the obstacle, (c) the string has both potential and kinetic energy and has wrapped around the obstacle. In a wrapped configuration, $(\bar{x}, \bar{y})$ denotes the coordinate where the string breaks contact with the obstacle.

A2. The string is homogenous and has a constant mass per unit length denoted by ρ. The tension in the string is equal to T and remains constant at all times. The string undergoes transverse vibration in the xy plane and is not affected by gravity.
A3. The amplitude of oscillation of the string is small and therefore the equation of motion of the string can be expressed by the standard relation [14]

$$\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{1}{c^2}\left(\frac{\partial^2 y}{\partial t^2}\right), \quad c \triangleq \sqrt{T/\rho} \tag{2}$$

where $y(x,t)$ is the displacement of the string at a distance x from the origin at time t.
A4. The string wraps around the obstacle during vibration. Over each time step during wrapping, a small element of the string comes to rest on the obstacle through perfectly inelastic collisions. The wrapping process continues till the freely vibrating portion of the string has no more kinetic energy.
A5. The surface of the obstacle is not sticky and the string unwraps from the obstacle without any loss of energy.
A6. At the initial time $t=0$, the string has no contact with the obstacle. It is in its mean position with zero potential energy and kinetic energy equal to E_0.
A7. The string continues to vibrate in the mode in which it started its vibration at the initial time. This implies that each point of the string, not in contact with the obstacle, has the same frequency of vibration[1] at any instant of time, and the number of nodes[2] in the vibrating string remains constant.
A8. The string has no internal damping, i.e., the energy of the string will remain conserved during free vibration.

3. Analytical model

3.1. Boundary conditions and general solution

A general solution to the partial differential equation in Eq. (2) can be written as [14]

$$y(x,t) = (\alpha_1 \sin\lambda x + \alpha_2 \cos\lambda x)(\alpha_3 \sin\omega t + \alpha_4 \cos\omega t) \tag{3}$$

where α_i, i=1, 2, 3, 4 are constants, ω is the circular frequency and λ is related to ω by the relation

$$\omega \triangleq c\lambda \tag{4}$$

At time t=0, the string is at the mean position, i.e., $y(x,0) \equiv 0$, per assumption A6. This implies $\alpha_4 = 0$. The solution in Eq. (3) can now be written as

$$y(x,t) = (A\sin\lambda x + B\cos\lambda x)\sin\omega t, \quad A = \alpha_1\alpha_3, \quad B = \alpha_2\alpha_3 \tag{5}$$

At the right boundary, the string satisfies the relation $y(l,t) = 0$ for all t. Using Eq. (5) we get

$$A\sin\lambda l + B\cos\lambda l = 0 \implies B = -A\tan\lambda l \tag{6}$$

Substitution of Eq. (6) into Eq. (5) gives the solution

$$y(x,t) = A(\sin\lambda x - \tan\lambda l \cos\lambda x)\sin\omega t \tag{7}$$

We now consider the boundary conditions at the contact break point. From Fig. 1 we have

$$f(\overline{x}) = y(\overline{x},t) \implies f(\overline{x}) = A(\sin\lambda\overline{x} - \tan\lambda l \cos\lambda\overline{x})\sin\omega t \tag{8}$$

Also, the string is tangential to the obstacle at the contact break point $x=\overline{x}$, i.e.,

$$f'(\overline{x}) = \frac{\partial y}{\partial x}(\overline{x},t) \implies f'(\overline{x}) = \lambda A(\cos\lambda\overline{x} + \tan\lambda l \sin\lambda\overline{x})\sin\omega t \tag{9}$$

From Eqs. (8) and (9) we get

$$\tan\lambda(l-\overline{x}) = -\lambda\frac{f(\overline{x})}{f'(\overline{x})} \tag{10}$$

which indicates that λ can be computed from the value of $\overline{x}$. The solution of Eq. (10) is, however, not unique—each non-trivial value of λ corresponds to a mode of vibration of the string. Since λ is an implicit function of $\overline{x}$, we can rewrite Eq. (9) as follows:

$$A\sin\omega t = g(\overline{x}), \quad g(x) \triangleq \frac{f'(x)}{\lambda(\cos\lambda x + \tan\lambda l \sin\lambda x)} \tag{11}$$

[1] The frequency of the string is not constant; it varies with time and amplitude when the string wraps and unwraps around the obstacle.
[2] A node is a point with zero displacement. As the string wraps or unwraps around the obstacle, the location of the node(s) change and therefore node(s) have non-zero horizontal velocity.

Eq. (11) can be used to compute t from the value of $\bar{x}$. The existence of the solution, however, depends on the magnitude of A. We now discuss the procedure for computing A.

Let the total energy of the string at any time t be denoted by E. Then,

$$E = E_{pe} + E_{ke} = E_{pe}^{obs} + E_{pe}^{vib} + E_{ke} \tag{12}$$

where E_{pe}^{obs} is the potential energy of the string wrapped around the obstacle, E_{pe}^{vib} is the potential energy of the freely vibrating string, E_{pe} is the total potential energy, and E_{ke} is the kinetic energy of the string. The total potential energy of the string is computed as the product of the tension T (which is assumed constant) and elongation of the string [15]. The elongation of the string is computed by integrating the strain of the string along the length wrapped around the obstacle and along the length of string vibrating freely. Thus, the total potential energy can be written as

$$E_{pe} = T\int \mathrm{d}l = T\int(\mathrm{d}s - \mathrm{d}x) = T\int\left(\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} - \mathrm{d}x\right) = T\int_0^l\left(\sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2} - 1\right)\mathrm{d}x$$

and E_{pe}^{obs} and E_{pe}^{vib} can be written as

$$E_{pe}^{obs} = T\int_0^{\bar{x}}\left(\sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2} - 1\right)\mathrm{d}x, \quad E_{pe}^{vib} = T\int_{\bar{x}}^{l}\left(\sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2} - 1\right)\mathrm{d}x$$

Since the string conforms to the shape of the obstacle, $(\mathrm{d}y/\mathrm{d}x) = f'(x)$ for $x \in [0, \bar{x}]$. By expressing $(\mathrm{d}y/\mathrm{d}x) = y'(x,t)$ for $x \in [\bar{x}, l]$ and simplifying using Eqs. (7) and (11), we get

$$E_{pe}^{obs} = T\int_0^{\bar{x}}\left[\sqrt{1 + [f'(x)]^2} - 1\right]\mathrm{d}x \tag{13}$$

$$\begin{aligned} E_{pe}^{vib} &= T\int_{\bar{x}}^{l}\left[\sqrt{1 + [y'(x,t)]^2} - 1\right]\mathrm{d}x \approx \frac{T}{2}\int_{\bar{x}}^{l}[y'(x,t)]^2\,\mathrm{d}x = \frac{T}{2}\lambda^2 A^2 \sin^2\omega t\int_{\bar{x}}^{l}[\cos\lambda x + \tan\lambda l \sin\lambda x]^2\,\mathrm{d}x \\ &= \frac{1}{8}T\lambda A^2 \sin^2\omega t \sec^2\lambda l\{2\lambda(l-\bar{x}) + \sin[2\lambda(l-\bar{x})]\} = \frac{1}{8}T\lambda\sec^2\lambda l\{2\lambda(l-\bar{x}) + \sin[2\lambda(l-\bar{x})]\}[g(\bar{x})]^2 \end{aligned} \tag{14}$$

An element of string of length $\mathrm{d}x$ has a mass of $\rho\,\mathrm{d}x$ and velocity is $\dot{y}(x,t)$. Thus, the kinetic energy of the freely vibrating string can be written and simplified as follows:

$$\begin{aligned} E_{ke} &= \frac{1}{2}\int_{\bar{x}}^{l}\rho[\dot{y}(x,t)]^2\,\mathrm{d}x = \frac{\rho}{2}\omega^2 A^2\cos^2\omega t\int_{\bar{x}}^{l}[\sin\lambda x - \tan\lambda l\cos\lambda x]^2\,\mathrm{d}x \\ &= \frac{1}{8\lambda}\rho\omega^2 A^2\cos^2\omega t\sec^2\lambda l\{2\lambda(l-\bar{x}) - \sin[2\lambda(l-\bar{x})]\} = \frac{1}{8\lambda}\rho\omega^2\sec^2\lambda l\{2\lambda(l-\bar{x}) - \sin[2\lambda(l-\bar{x})]\}\{A^2 - [g(\bar{x})]^2\} \end{aligned} \tag{15}$$

From Eqs. (12) to (15) it is easy to verify that the energy expression has the form

$$E = h(\bar{x}, A) \tag{16}$$

For a configuration in which the string is wrapped around the obstacle, the complete solution can be determined from the values of $\bar{x}$ and E using the four-step algorithm below:

1. Use Eq. (10) to determine the value of λ. Since Eq. (10) provides multiple non-trivial solutions that correspond to different modes of vibration, the solution corresponding to the initial mode of vibration should be chosen—see assumption A7.
2. Use Eq. (4) to compute ω.
3. Compute A from Eq. (16) using the values of $\bar{x}$, E, λ and ω.
4. Compute the time t from Eq. (11) by substituting in the values of $\bar{x}$, A and λ.

The complete solution can now be described using Eqs. (1) and (7) as follows:

$$y(x,t) = \begin{cases} f(x), & x \in [0, \bar{x}] \\ A(\sin\lambda x - \tan\lambda l\cos\lambda x)\sin\omega t, & x \in [\bar{x}, l] \end{cases} \tag{17}$$

3.2. Wrapping of the string

From our discussion in the last section we know that the geometry of the string can be determined from the values of $\bar{x}$ and E. In this section we discuss the method for computing these values at regular intervals of time. Let $\{\bar{x}_i, E_i\}$ denote the values of $\bar{x}$ and E at time $t = t_i$, $i = 0, 1, 2, \ldots, k$. We assume $t_0 = 0$. Then, from assumption A6, $\bar{x}_0 = 0$ and the value of E_0 is known. We will discuss the method for determining the value of t_k which denotes the time after which the string begins to unwrap.

Let us assume that for some i=j, $\{\bar{x}_j, E_j\}$ is known. We outline the method for computing $\{\bar{x}_{j+1}, E_{j+1}\}$ from the values of $\{\bar{x}_j, E_j\}$. Choose a small segment of the vibrating string that is expected to wrap around the obstacle over a small interval of

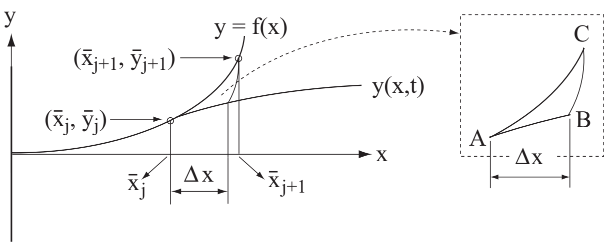

Fig. 2. Small string segment Δx wraps around obstacle after perfectly inelastic collision. In the magnified image, AB denotes the small string segment of length Δx that wraps around the obstacle over the region AC.

time. Let the projection of this string segment AB on the x-axis be Δx as shown in Fig. 2. The kinetic energy of this string segment, which will be lost due to inelastic collision, can be computed from Eq. (7) as follows:

$$E_{\text{lost}} = \frac{\rho}{2}\int_{\bar{x}_j}^{\bar{x}_j+\Delta x} [\dot{y}(x,t)]^2\,\mathrm{d}x = \frac{\rho}{2}\omega_j^2 A_j^2 \cos^2 \omega_j t_j \int_{\bar{x}_j}^{\bar{x}_j+\Delta x} \left[\sin\lambda_j x - \tan\lambda_j l \cos\lambda_j x\right]^2 \mathrm{d}x \tag{18}$$

where A_j, ω_j, λ_j and t_j denote values of A, ω, λ and t, respectively, derived for the pair $\{\bar{x}_j, E_j\}$. Using Eq. (18), E_{j+1} can be computed as follows:

$$E_{j+1} = E_j - E_{\text{lost}}, \quad j = 1,2,\ldots,k-1 \tag{19}$$

To compute $\bar{x}_{j+1}$, $j = 1,2,\ldots,k-1$, we make the following general assumption:

A9. With reference to Fig. 2, the potential energy of the vibrating string segment AB at time t_j is equal to the potential energy of the string segment AC wrapped on the obstacle at time t_{j+1}.

Using Eqs. (13) and (14) assumption A9 can be mathematically expressed as follows:

$$\int_{\bar{x}_j}^{\bar{x}_{j+1}} \left[\sqrt{1+[f'(x)]^2}-1\right]\mathrm{d}x = \int_{\bar{x}_j}^{\bar{x}_j+\Delta x} \left[\sqrt{1+[y'(x,t)]^2}-1\right]\mathrm{d}x \approx \frac{1}{2}\int_{\bar{x}_j}^{\bar{x}_j+\Delta x} [y'(x,t)]^2\,\mathrm{d}x$$
$$= \frac{1}{2}\lambda_j^2 A_j^2 \sin^2\omega_j t_j \int_{\bar{x}_j}^{\bar{x}_j+\Delta x} [\cos\lambda_j x + \tan\lambda_j l \sin\lambda_j x]^2\,\mathrm{d}x \tag{20}$$

Eq. (20) can be used to determine $\bar{x}_{j+1}$. The values of A_{j+1}, ω_{j+1}, λ_{j+1} and t_{j+1} are computed from the values of $\bar{x}_{j+1}$ and E_{j+1}. The iterative process is terminated when the kinetic energy of the vibrating string segment becomes approximately equal to zero. At this time, which is denoted as t_k, the string stops wrapping and begins to unwrap.

3.3. *Unwrapping of the string*

Similar to wrapping, the geometry of the string during unwrapping is computed from the values of $\bar{x}$ and E. The string begins to unwrap at $t = t_k$; at this time the values of $\bar{x} = \bar{x}_k$ and $E = E_k$ are known. Let $\{\bar{x}_i, E_i\}$ denote the values of $\bar{x}$ and E at time $t = t_i$, $i = k, k+1, k+2, \ldots, l$, where t_l denotes the time when the string has unwrapped completely. We outline the method for computing $\{\bar{x}_{j+1}, E_{j+1}\}$ from the values of $\{\bar{x}_j, E_j\}$ for $k \le j \le l-1$. One chooses a small segment of the string that is expected to unwrap over a small interval of time. Then we let the projection of this string segment on the x-axis be Δx. Then,

$$\bar{x}_{j+1} = \bar{x}_j - \Delta x, \quad j = k, k+1, \ldots, l-1 \tag{21}$$

Since there is no loss of kinetic energy during unwrapping (see assumption A5), we have

$$E_{j+1} = E_j, \quad j = k, k+1, \ldots, l-1 \tag{22}$$

The values of A_{j+1}, ω_{j+1}, λ_{j+1} and t_{j+1} are computed iteratively from the values of $\bar{x}_{j+1}$ and E_{j+1}. The iterative process is terminated at $t = t_l$ when the potential energy of the string is equal to its value at the mean position.

4. Numerical simulations

Consider a string with

$$T = 1\,\text{N}, \quad \rho = 0.025\,\text{kg/m}, \quad l = 4\,\text{m} \tag{23}$$

The obstacle is assumed to be a circle of radius R and center coordinates $(x, y) \equiv (0, R)$, i.e.,

$$y = f(x) = R - \sqrt{R^2 - x^2}, \quad 0 \le x \le R \tag{24}$$

Table 1
Percentage energy loss over one cycle of oscillation and $\overline{x}_k$ for different values of E_0 and three modes of oscillation, all with R=1 m.

	Mode 1	Mode 2	Mode 3
E_0=1.00 J	0.491%, 0.729 m	1.598%, 0.696 m	2.752%, 0.654 m
E_0= 0.50 J	0.256%, 0.596 m	0.861%, 0.569 m	1.547%, 0.537 m
E_0=0.25 J	0.113%, 0.461 m	0.402%, 0.445 m	0.766%, 0.424 m

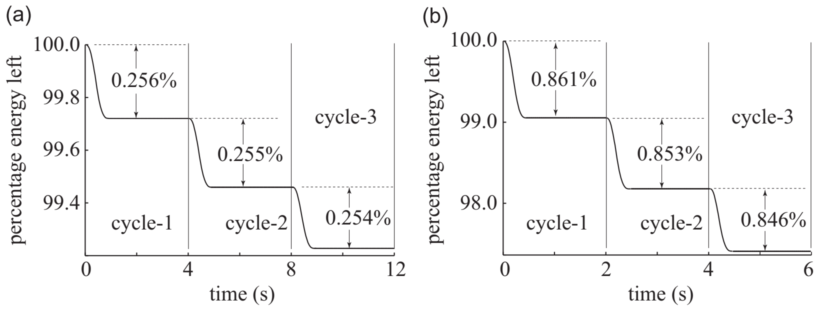

Fig. 3. Plot of percentage energy content of the string over three consecutive cycles of vibration in (a) Mode 1, and (b) Mode 2. For both cases, the initial energy of the string was E_0=0.5 J.

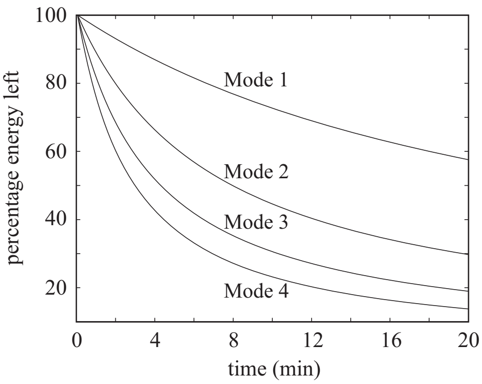

Fig. 4. Exponential decay in the energy of a string wrapping and unwrapping around an obstacle. The plots show energy decay for single-mode vibration in the first four modes with E_0=0.5 J.

It can be verified that $f(x)$ in Eq. (24) satisfies the boundary conditions in Eq. (1). For R=1 m and $\Delta x = 0.001$ m, we compute the percentage loss of energy over one cycle of string oscillation for three different values of initial energy E_0 and for oscillation in the first, second, and third modes, respectively. These values are shown in Table 1 together with the values of $\overline{x}_k$, which is a measure of the length of wrapping around the obstacle. For the special case of E_0=0.5 J, we plot the percentage loss of energy for three consecutive cycles of string vibration in the first two modes. These plots are shown in Fig. 3. Fig. 4 plots the decay in energy as a function of time for vibration in the first four modes with E_0=0.5 J. The following observations can be made from the plots in Figs. 3 and 4, and the data in Table 1:

- For any mode of oscillation, it can be seen that the percentage energy loss is higher for higher values of E_0. This is not surprising since higher values of E_0 results in higher kinetic energy and greater length of wrapping, as evident from the values of $\overline{x}_k$ in Table 1, and consequently more energy loss through inelastic collision. The same argument can explain the reduction in the percentage loss of energy over consecutive cycles of vibration in Fig. 3.
- The percentage energy loss is higher for higher modes of oscillation for the same value of E_0. This is true for the same number of cycles (see Fig. 3) as well as for the same length of time (see Fig. 4) and is due to the fact that the velocities of the string associated with higher frequencies are higher in higher modes, and as a consequence the loss upon impact is higher. The value of $\overline{x}_k$ is less for the higher modes but this does not have a significant effect on the percentage of energy loss.

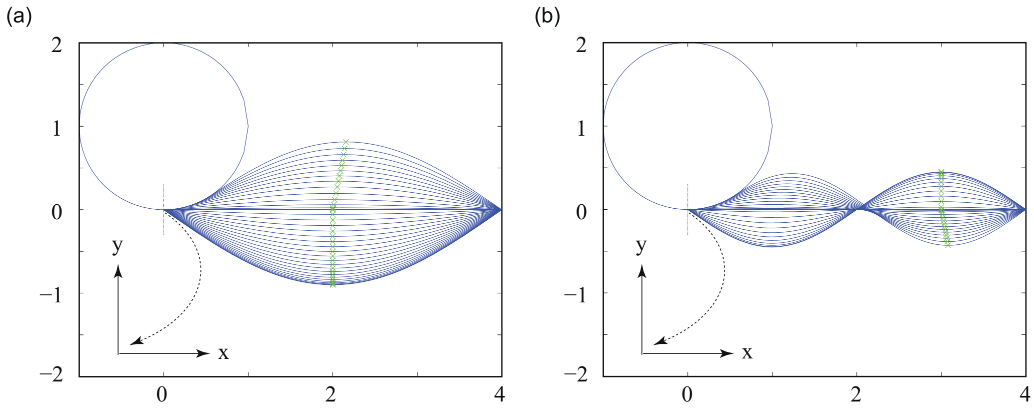

Fig. 5. A string vibrating against a circular obstacle in (a) Mode 1, and (b) Mode 2.

Table 2
Percentage energy loss over one cycle of oscillation and $\bar{x}_k$ for obstacles of different shapes and sizes, all with E_0=0.50 J.

	Mode 1	Mode 2	Mode 3
Case (a)	0.029%, 0.295 m	0.113%, 0.291 m	0.237%, 0.286 m
Case (b)	0.256%, 0.596 m	0.861%, 0.569 m	1.547%, 0.537 m
Case (c)	0.915%, 0.897 m	2.549%, 0.815 m	3.887%, 0.737 m
Case (d)	0.665%, 0.803 m	2.021%, 0.750 m	3.278%, 0.692 m

The geometry of the string at different points in time during one cycle of oscillation is shown in Fig. 5 for Mode 1 and Mode 2 with initial energy E_0=0.5 J. It can be seen from these plots that a fixed point on the string moves in the y direction only when the string is not in contact with the obstacle but moves in both the x and y directions during wrapping and unwrapping. From the plot for Mode 2, it is also clear that a node is not a fixed point on the string. It is a point of zero displacement but has non-zero velocity during wrapping and unwrapping.

To study the effect of the shape of the obstacle on percentage energy loss, we fix the value of the initial energy to E_0=0.5 J and study the following four cases where the obstacle is:

(a) a circle with R=0.5 m;
(b) a circle with R=1.0 m;
(c) a circle with R=1.5 m;
(d) an ellipse with semi-major and semi-minor axes lengths of 1.2 and 1.0 m, respectively, and with the major axis aligned with the x-axis;

and satisfy the boundary conditions in Eq. (1). The results are shown in Table 2. It is clear from the results that for circular obstacles the percentage energy loss increases with increase in radius and vice versa. This is in agreement with the results expected for the limiting cases, namely, percentage energy loss is zero when the radius of the circle is zero and is equal to 100 percent when the radius is infinity. The ellipse in case (d) circumscribes the circle in case (b) and provides a lower slope for the wrapping curve. A comparison of the data for cases (b) and (d) indicates that a slight decrease in slope of the obstacle results in significantly higher percentage of energy loss.

5. Effect of change in slope of obstacle

We consider the obstacle in Fig. 6 where the curve y=$g(x)$ is obtained by rotating the curve y=$f(x)$ in Fig. 1 clockwise by angle θ about point O. To deal with this problem, we modify assumptions A1 and A6 as follows:

A1. The obstacle is rigid and has the following geometry:

$$y = f(x), \quad y(0) = 0, \quad \left[\frac{dy}{dx}\right]_{x=0} = -\tan\theta \tag{25}$$

A6. At the initial time t=0, the string has no contact with the obstacle. It has zero kinetic energy and potential energy equal to E_0. The displacement of the string at the initial time corresponds to a single mode of free vibration as shown in Fig. 6.

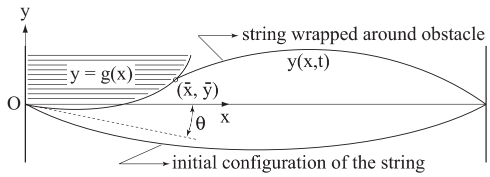

Fig. 6. A string vibrating against an obstacle. The obstacle is identical to the one in Fig. 1 but rotated clockwise by angle θ about point O.

Table 3
Percentage energy loss over one cycle of oscillation and $\bar{x}_k$ for two modes of oscillation with different values of θ.

θ (deg)	Mode 1	Mode 2
0	0.256%, 0.596 m	0.113%, 0.291 m
15	1.486%, 0.866 m	3.478%, 0.841 m
30	9.233%, 1.095 m	13.72%, 1.080 m

The remaining assumptions, A2–A5 and A6–A9, are not changed. In Table 3 we present simulation results for a string with

$$T = 1\,\text{N}, \quad \rho = 0.025\,\text{kg/m}, \quad l = 4\,\text{m}, \quad E_0 = 0.50\,\text{J} \tag{26}$$

and a circular obstacle of radius R=1 m. A comparison of the results indicates that percentage energy loss is significantly higher for higher values of θ.

In our analysis, the string was assumed to have no damping. In reality, the string will have damping and this will enhance the rate of energy decay. The rate of energy decay will, however, not be constant even if the damping ratio of the string is constant. As the string wraps around the obstacle, its effective length decreases and frequency of vibration increases—this will increase the rate of energy decay which depends on the product of damping ratio and natural frequency.

6. Conclusion

The vibration of a string wrapping and unwrapping around a smooth obstacle was investigated in this paper. Assuming linear behavior of the string, an analytical model was developed for computing its geometry at each time step by bookkeeping the energy. The energy of the string is assumed to dissipate during wrapping through inelastic collision between the string and the obstacle but remain conserved during unwrapping. The obstacle serves as a passive mechanism for damping and its effectiveness can be increased by changing its orientation in a manner that results in greater wrapping. This leads to the possibility of rapid dissipation of vibration energy through active control of the orientation of the obstacle and extension of the methodology for active vibration control of soft structures such as thin plates and membranes. Such work, however, lies in the scope of our future research.

References

[1] C. Citrini, The energy theorem in the impact of a string vibrating against a pointshaped obstacle, *Atti della Accademia Nazionale dei Lincei Rendiconti* 62 (1977) 143–148.
[2] L. Amerio, Continuous solutions of the problem of a string vibrating against an obstacle, *Rendiconti del Seminario Mathematico della Universita di Padova* 59 (1978) 67–96.
[3] M. Schatzman, A hyperbolic problem of second order with unilateral constraints: the vibrating string with a concave obstacle, *Journal of Mathematical Analysis and Applications* 73 (1980) 138–191.
[4] A. Haraux, H. Cabannes, Almost periodic motion of a string vibrating against a straight fixed obstacle, *Nonlinear Analysis, Theory, Methods and Applications* 7 (2) (1983) 129–141.
[5] R. Burridge, J. Kappraff, C. Morshedi, The sitar string, a vibrating string with a one-sided inelastic constraint, *SIAM Journal of Applied Mathematics* 42 (6) (1982) 1231–1251.
[6] A. Bamberger, M. Schatzman, New results on the vibrating string with a continuous obstacle, *SIAM Journal of Mathematical Analysis* 14 (3) (1983) 560–595.
[7] J. Ahn, A vibrating string with frictionless dynamic impact, *Applied Numerical Mathematics* 57 (2007) 861–884.
[8] M. Frontini, L. Gotusso, Numerical study of the motion of a string vibrating against an obstacle by physical discretization, *Applied Mathematical Modeling* 14 (1990) 489–494.
[9] A. Krishnaswamy, J.O. Smith, Methods for simulating string collisions with rigid spatial obstacles, *IEEE Workshop on Applications of Signal Processing to Audio and Acoustics*, New Paltz, NY, 2003, pp. 233–236.
[10] S.M. Han, M.A. Grosenbaugh, Nonlinear free vibration of a cable against a straight obstacle, *Journal of Sound and Vibration* 273 (2004) 337–361.

[11] J. Issa, R. Mukherjee, A.R. Diaz, Energy dissipation in dynamical systems through sequential application and removal of constraints, *ASME Journal of Dynamic Systems, Measurement and Control* 131 (2) (2009).
[12] W. Liu, Optimal boundary obstacle of the string vibration, *IEEE International Conference on Control Applications*, Anchorage, Alaska, 2000, pp. 959–964.
[13] S.M. Shahruz, C.A. Narasimha, Suppression of vibration in stretched strings by the boundary control, *Journal of Sound and Vibration* 204 (5) (1997) 835–840.
[14] E. Kreyzig, *Advanced Engineering Mathematics*, ninth ed., Wiley, Hoboken, NJ.
[15] L. Meirovitch, *Elements of Vibration Analysis*, McGraw-Hill, New York, NY.